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Torsion units for a Ree group, Tits group and a Steinberg triality group

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Abstract We investigate the Zassenhaus conjecture for the Steinberg triality group ${}^3D_4(2^3)$, Tits group ${}^2F_4(2)'$ and the Ree group ${}^2F_4(2)$. Consequently, we prove that the Prime Graph question is true for all three groups.

Keywords Zassenhaus conjecture · Prime graph question · Torsion unit · Partial augmentation · Integral group ring

Mathematics Subject Classification 16S34 · 20C05

1 Introduction and main results

Let $U(\mathbb{Z}G)$ be the unit group of the integral group ring of a finite group G. It is well known that

$$U(\mathbb{Z}G) = \{\pm 1\} \times V(\mathbb{Z}G),$$

where $V(\mathbb{Z}G)$ is the group of units of augmentation one. Throughout this paper, G will always represent a finite group and torsion units will always represent torsion units in $V(\mathbb{Z}G)\backslash\{1\}$. The next conjecture is very important in theory of integral group rings:

Conjecture 1 *If G is a finite group, then for each torsion unit $u \in V(\mathbb{Z}G)$ there exists $g \in G$, such that $|u| = |g|$.*

Hans Zassenhaus formulated a stronger version of this conjecture in [1], which states:

Conjecture 2 *A torsion unit in $V(\mathbb{Z}G)$ is said to be rationally conjugate to a group element if it is conjugate to an element of G by a unit of the rational group ring $\mathbb{Q}G$.*

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This conjecture was confirmed for Nilpotent Groups in [2,3] and cyclic-by-abelian groups in [4]. However, these techniques do not transfer to simple groups. The Luthar–Passi Method (which was introduced in [5]) is the main investigative tool for simple groups in relation to the Zassenhaus conjecture. The Zassenhaus conjecture was confirmed true for all groups up to order 71 in [6]. This conjecture was also validated for A_5, S_5, central extensions of S_5 and other simple finite groups in [5,7–9]. In [10] partial results were given for A_6, and the remaining cases were dealt with in [11]. Alternating groups of higher order were also considered in [12,13]. Additionally, it was proved for $PSL(2, p)$ when $p = \{7, 11, 13\}$ in [14] and $p = \{8, 17\}$ in [15] and $PSL(2, p)$ when $p = \{19, 23\}$ in [16]. See [17,18] for further results regarding $PSL(2, p)$.

Let H be a group with a torsion part $t(H)$ of finite exponent and $\#H$ be the set of primes dividing the order of elements from the set $t(H)$. The prime graph of H (denoted by $\pi(H)$) is a graph with vertices labeled by primes from $\#H$, such that vertices p and q are adjacent if and only if there is an element of order pq in the group H. The following, was composed as a problem in [19] (Problem 37):

Question 1 (Prime Graph Question) *If G is a finite group, then $\pi(G) = \pi(V(\mathbb{Z}G))$.*

This question is upheld for Frobenius and Solvable groups in [20]. It was also confirmed for some Sporadic Simple groups in [21–32]. Additionally, it was confirmed for the sympletic simple group $S_4(4)$ in [33]. The prime graph question is an intermediate step towards the Zassenhaus conjecture.

The family of Steinberg Triality groups (denoted by ${}^3D_4(q^3)$) were introduced by Steinberg in [34]. The Ree groups ${}^2F_4(2^{n+1})$ were initially constructed in [35]. These groups are simple if $n \geq 1$ and Tits constructed a new simple group from the derived subgroup of ${}^2F_4(2)$ (denoted by ${}^2F_4(2)'$) in [36].

We combine the Luthar–Passi Method together with techniques developed in [14,37] to obtain our results. Our main results are as follows:

Theorem 1 *Let $G = {}^3D_4(2^3)$ and u be a torsion unit of $V(\mathbb{Z}G)$. Subsequently the following conditions hold:*

(i) *If $|u| \in \{7\}$, then u is rationally conjugate to some g in G.*
(ii) *There are no elements of orders 26, 39 or 91 in $V(\mathbb{Z}G)$.*
(iii) *If $|u| = 2$, then $\nu_{rx} = 0 \,\forall\, rx \notin \{\nu_{2a}, \nu_{2b}\}$ and*

$$(\nu_{2a}, \nu_{2b}) \in \{(2, -1), (1, 0), (0, 1), (-1, 2), (-2, 3)\}.$$

(iv) *If $|u| = 3$, then $\nu_{rx} = 0 \,\forall\, rx \notin \{\nu_{3a}, \nu_{3b}\}$ and*

$$(\nu_{3a}, \nu_{3b}) \in \{(3, -2), (2, -1), (1, 0), (0, 1), (-1, 2)\}.$$

(v) *If $|u| = 13$, then $\nu_{rx} = 0 \,\forall\, rx \notin \{\nu_{13a}, \nu_{13b}, \nu_{13c}\}$ and*

$$(\nu_{13a}, \nu_{13b}, \nu_{13c}) \in \{(1, -1, 1), (1, 0, 0), (0, 0, 1), (1, 1, -1), (0, 1, 0), (-1, 1, 1)\}.$$

Theorem 2 *Let $G = {}^2F_4(2)'$ and u be a torsion unit of $V(\mathbb{Z}G)$. Subsequently the following conditions hold:*

(i) *If $|u| \in \{3, 5\}$, then u is rationally conjugate to some g in G.*
(ii) *There are no elements of orders 15, 26, 39 or 65 in $V(\mathbb{Z}G)$.*
(iii) *If $|u| = 2$, then $\nu_{rx} = 0 \,\forall\, rx \notin \{\nu_{2a}, \nu_{2b}\}$ and*

$$(\nu_{2a}, \nu_{2b}) \in \{(3, -2), (2, -1), (1, 0), (0, 1), (-1, 2), (-2, 3), (-3, 4)\}.$$

(iv) *If* $|u| = 10$*, then* $\nu_{rx} = 0 \,\forall\, rx \notin \{\nu_{2a}, \nu_{2b}, \nu_{5a}, \nu_{10a}\}$ *and*

$$\begin{aligned}(\nu_{2a}, \nu_{2b}, \nu_{5a}, \nu_{10a}) \in \{&(-2,-2,0,5), (-1,-3,0,5), (-1,-3,5,0), (-1,-1,0,3),\\ &(-1,-1,5,-2), (-1,1,0,1), (0,-4,5,0), (0,-2,0,3),\\ &(0,-2,5,-2), (0,0,0,1), (0,0,5,-4), (0,2,0,-1),\\ &(0,4,-5,2), (0,4,0,-3), (1,1,0,-1), (1,3,-5,2),\\ &(1,3,0,-3), (2,2,0,-3)\}.\end{aligned}$$

(v) *If* $|u| = 13$*, then* $\nu_{rx} = 0 \,\forall\, rx \notin \{\nu_{13a}, \nu_{13b}\}$ *and*

$$\begin{aligned}(\nu_{13a}, \nu_{13b}) \in \{&(9,-8), (8,-7), (7,-6), (6,-5), (5,-4), (4,-3), (3,-2), (2,-1),\\ &(1,0), (0,1), (-1,2), (-2,3), (-3,4), (-4,5), (-5,6), (-6,7),\\ &(-7,8), (-8,9)\}.\end{aligned}$$

Theorem 3 *Let* $G = {}^2F_4(2)$ *and u be a torsion unit of* $V(\mathbb{Z}G)$*. Subsequently the following conditions hold:*

(i) *If* $|u| \in \{3, 5, 13\}$*, then u is rationally conjugate to some g in G.*
(ii) *There are no elements of orders* 15*,* 26*,* 39 *or* 65 *in* $V(\mathbb{Z}G)$*.*
(iii) *If* $|u| = 2$*, then* $\nu_{rx} = 0 \,\forall\, rx \notin \{\nu_{2a}, \nu_{2b}\}$ *and*

$$(\nu_{2a}, \nu_{2b}) \in \{(3,-2), (2,-1), (1,0), (0,1), (-1,2), (-2,3), (-3,4)\}.$$

(iv) *If* $|u| = 10$*, then* $\nu_{rx} = 0 \,\forall\, rx \notin \{\nu_{2a}, \nu_{2b}, \nu_{5a}, \nu_{10a}\}$ *and*

$$\begin{aligned}(\nu_{2a}, \nu_{2b}, \nu_{5a}, \nu_{10a}) \in \{&(-2,-2,0,5), (-1,-3,0,5), (-1,-3,5,0), (-1,-1,0,3),\\ &(-1,-1,5,-2), (-1,1,0,1), (0,-4,5,0), (0,-2,0,3),\\ &(0,-2,5,-2), (0,0,0,1), (0,0,5,-4), (0,2,0,-1),\\ &(0,4,-5,2), (0,4,0,-3), (1,1,0,-1), (1,3,-5,2),\\ &(1,3,0,-3), (2,2,0,-3)\}.\end{aligned}$$

Corollary 1 *The Prime Graph question is true for the groups* ${}^3D_4(2)$*,* ${}^2F_4(2)'$ *and* ${}^2F_4(2)$*.*

Let $u = \sum a_g g$ be a torsion unit of $V(\mathbb{Z}G)$. Then, the sum $\sum_{g \in X^G} a_g \in \mathbb{Z}$ is the partial augmentation (denoted by $\varepsilon_C(u)$) of u with respect to it's conjugacy classes X^G in G. Let $\nu_i = \varepsilon_{C_i}(u)$ be the i-th partial augmentation of u. It was proved that $\nu_1 = 0$ and $\nu_j = 0$ if the conjugacy class C_j consists of a central element by G. Higman and S. D. Berman [38]. Therefore, $\nu_2 + \nu_3 + \cdots + \nu_l = 1$, where l denotes the number of non-central conjugacy classes of G.

Proposition 1 ([39]) *Let u be torsion unit of* $V(\mathbb{Z}G)$*. The order of u divides the exponent of* G*.*

The following propositions provide relationships between the partial augmentations and the order of a torsion unit.

Proposition 2 (Proposition 3.1 in [37]) *Let u be a torsion unit of* $V(\mathbb{Z}G)$*. Let C be a conjugacy class of G. If p is a prime dividing the order of a representative of C but not the order of u then the partial augmentation* $\varepsilon_C(u) = 0$*.*

Proposition 3 (Proposition 2.2 in [14]) *Let G be a finite group and let u be a torsion unit in* $V(\mathbb{Z}G)$*.*

(i) *If u has order p^n, then $\varepsilon_x(u) = 0$ for every x of G whose p-part is of order strictly greater than p^n.*
(ii) *If x is an element of G whose p-part, for some prime, has order strictly greater than the order of the p-part of u, then $\varepsilon_x(u) = 0$.*

Proposition 4 ([5]) *Let u be a torsion unit of $V(\mathbb{Z}G)$ of order k. Then u is conjugate in $\mathbb{Q}G$ to an element $g \in G$ iff for each d dividing k there is precisely one conjugacy class νi_d with partial augmentation $\varepsilon_{\nu i_d}(u^d) \neq 0$.*

Proposition 5 ([5]) *Let p be equal to zero or a prime divisor of $|G|$. Suppose that u is an element of $V(\mathbb{Z}G)$ of order k. Let z be a primitive kth root of unity. Then for every integer l and any character χ of G, the number*

$$\mu_l(u, \chi, p) = \frac{1}{k} \sum_{d|k} Tr_{\mathbb{Q}(z^d)/\mathbb{Q}} \left\{ \chi(u^d) z^{-dl} \right\}$$

is a non-negative integer.

We will use the notation $\mu_l(u, \chi, *)$ when $p = 0$. The LAGUNA package [40] for the GAP system [41] is a very useful tool when calculating $\mu_l(u, \chi, p)$.

2 Proof of Theorem 1

Let $G = {}^3D_4(2^3)$. Clearly $|G| = 211341312 = 2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$ and $exp(G) = 6552 = 2^3 \cdot 3^2 \cdot 7 \cdot 13$. Initially, for any torsion unit of $V(\mathbb{Z}G)$ of order k:

$$\begin{aligned}\nu_{2a} + \nu_{2b} + \nu_{3a} + \nu_{3b} + \nu_{4a} + \nu_{4b} + \nu_{4c} + \nu_{6a} + \nu_{6b} + \nu_{7a} + \nu_{7b} + \nu_{7c} + \nu_{7d} + \nu_{8a} \\ + \nu_{8b} + \nu_{9a} + \nu_{9b} + \nu_{9c} + \nu_{12a} + \nu_{13a} + \nu_{13b} + \nu_{13c} + \nu_{14a} + \nu_{14b} + \nu_{14c} \\ + \nu_{18a} + \nu_{18b} + \nu_{18c} + \nu_{21a} + \nu_{21b} + \nu_{21c} + \nu_{28a} + \nu_{28b} + \nu_{28c} = 1.\end{aligned}$$

In order to prove that the Zassenhaus Conjecture holds, we need to consider torsion units of $V(\mathbb{Z}G)$ of order 2, 3, 4, 6, 7, 8, 9, 12, 13, 14, 18, 21, 28, 24, 26, 36, 39, 42, 56, 63 and 91 (by Proposition 1). For the purpose of this paper and due to the complexity of certain orders, we shall consider elements of order 2, 3, 7, 13, 26, 39 and 91. We shall now consider each case separately.

Case (i). Let $u \in V(\mathbb{Z}G)$ where $|u| = 2$. Using Propositions 2 and 3,

$$\nu_{2a} + \nu_{2b} = 1.$$

Applying Proposition 5, we obtain the following system of inequalities:

$$\begin{aligned}\mu_0(u, \chi_2, *) &= \tfrac{1}{2}(-2\gamma_1 + 26) \geq 0; \quad \mu_1(u, \chi_2, *) = \tfrac{1}{2}(2\gamma_1 + 26) \geq 0; \\ \mu_0(u, \chi_3, *) &= \tfrac{1}{2}(4\gamma_2 + 52) \geq 0; \quad \mu_1(u, \chi_3, *) = \tfrac{1}{2}(-4\gamma_2 + 52) \geq 0\end{aligned}$$

where $\gamma_1 = 3\nu_{2a} - \nu_{2b}$ and $\gamma_2 = 5\nu_{2a} - \nu_{2b}$. It follows that the only possible integer solutions for (ν_{2a}, ν_{2b}) are listed in part (iii) of Theorem 1.

Case (ii). Let $u \in V(\mathbb{Z}G)$ where $|u| = 3$. Using Propositions 2 and 3,

$$\nu_{3a} + \nu_{3b} = 1.$$

Applying Proposition 5, we obtain the following system of inequalities:

$$\mu_0(u,\chi_3,*)=\tfrac{1}{3}(2\gamma_1+52)\geq 0;\quad \mu_1(u,\chi_3,*)=\tfrac{1}{3}(-\gamma_1+52)\geq 0;$$
$$\mu_0(u,\chi_2,2)=\tfrac{1}{3}(2\gamma_2+8)\geq 0;\quad \mu_1(u,\chi_2,2)=\tfrac{1}{3}(-\gamma_2+8)\geq 0$$

where $\gamma_1 = 7\nu_{3a} - 2\nu_{3b}$ and $\gamma_2 = 2\nu_{3a} - \nu_{3b}$. It follows that the only possible integer solutions for (ν_{2a}, ν_{2b}) are listed in part (iv) of Theorem 1.

Case (iii). Let $u \in V(\mathbb{Z}G)$ where $|u| = 7$. Using Propositions 2 and 3,

$$\nu_{7a}+\nu_{7b}+\nu_{7c}+\nu_{7d}=1.$$

Applying Proposition 5, we obtain:

$$\mu_0(u,\chi_2,*)=\tfrac{1}{7}(6\gamma_1+26)\geq 0;\quad \mu_1(u,\chi_2,*)=\tfrac{1}{7}(-\gamma_1+26)\geq 0;$$
$$\mu_0(u,\chi_5,*)=\tfrac{1}{7}(42\gamma_2+273)\geq 0;\quad \mu_1(u,\chi_5,*)=\tfrac{1}{7}(-7\gamma_2+273)\geq 0;$$
$$\mu_0(u,\chi_6,*)=\tfrac{1}{7}(6\gamma_3+324)\geq 0;\quad \mu_1(u,\chi_6,*)=\tfrac{1}{7}(-\gamma_3+324)\geq 0;$$
$$\mu_0(u,\chi_{10},*)=\tfrac{1}{7}(6\gamma_4+468)\geq 0;\quad \mu_1(u,\chi_{10},*)=\tfrac{1}{7}(-\gamma_4+468)\geq 0;$$
$$\mu_0(u,\chi_6,13)=\tfrac{1}{7}(6\gamma_5+323)\geq 0;\quad \mu_1(u,\chi_6,13)=\tfrac{1}{7}(-\gamma_5+323)\geq 0;$$
$$\mu_0(u,\chi_2,3)=\tfrac{1}{7}(4\gamma_6+25)\geq 0;\quad \mu_1(u,\chi_2,3)=\tfrac{1}{7}(-\gamma_6+25)\geq 0;$$
$$\mu_0(u,\chi_2,2)=\tfrac{1}{7}(-2\gamma_6+8)\geq 0;\quad \mu_0(u,\chi_{12},*)=\tfrac{1}{7}(6\gamma_7+1053)\geq 0;$$
$$\mu_1(u,\chi_{12},*)=\tfrac{1}{7}(-\gamma_7+1053)\geq 0;\quad \mu_1(u,\chi_2,2)=\tfrac{1}{7}(\gamma_8+8)\geq 0;$$
$$\mu_3(u,\chi_2,2)=\tfrac{1}{7}(\gamma_9+8)\geq 0;\quad \mu_2(u,\chi_2,2)=\tfrac{1}{7}(\gamma_{10}+8)\geq 0;$$
$$\mu_1(u,\chi_{18},*)=\tfrac{1}{7}(\gamma_{11}+2106)\geq 0;\quad \mu_2(u,\chi_{18},*)=\tfrac{1}{7}(\gamma_{12}+2106)\geq 0;$$
$$\mu_3(u,\chi_{18},*)=\tfrac{1}{7}(\gamma_{13}+2106)\geq 0;\quad \mu_0(u,\chi_{25},*)=\tfrac{1}{7}(\gamma_{14}+2808)\geq 0;$$
$$\mu_1(u,\chi_{25},*)=\tfrac{1}{7}(\gamma_{15}+2808)\geq 0;\quad \mu_2(u,\chi_{25},*)=\tfrac{1}{7}(\gamma_{16}+2808)\geq 0;$$
$$\mu_3(u,\chi_{25},*)=\tfrac{1}{7}(\gamma_{17}+2808)\geq 0;\quad \mu_1(u,\chi_6,2)=\tfrac{1}{7}(\gamma_{18}+48)\geq 0;$$
$$\mu_2(u,\chi_6,2)=\tfrac{1}{7}(\gamma_{19}+48)\geq 0;\quad \mu_3(u,\chi_6,2)=\tfrac{1}{7}(\gamma_{20}+48)\geq 0;$$
$$\mu_0(u,\chi_9,2)=\tfrac{1}{7}(\gamma_{21}+160)\geq 0;\quad \mu_1(u,\chi_9,2)=\tfrac{1}{7}(\gamma_{22}+160)\geq 0;$$
$$\mu_2(u,\chi_9,2)=\tfrac{1}{7}(\gamma_{23}+160)\geq 0;\quad \mu_3(u,\chi_9,2)=\tfrac{1}{7}(\gamma_{23}+160)\geq 0;$$
$$\mu_1(u,\chi_{13},2)=\tfrac{1}{7}(\gamma_{25}+784)\geq 0;\quad \mu_2(u,\chi_{13},2)=\tfrac{1}{7}(\gamma_{26}+784)\geq 0;$$
$$\mu_3(u,\chi_{13},2)=\tfrac{1}{7}(\gamma_{27}+784)\geq 0;\quad \mu_1(u,\chi_{22},*)=\tfrac{1}{7}(\gamma_{28}+2457)\geq 0;$$
$$\mu_2(u,\chi_{22},*)=\tfrac{1}{7}(\gamma_{29}+2457)\geq 0;\quad \mu_3(u,\chi_{22},*)=\tfrac{1}{7}(\gamma_{30}+2457)\geq 0$$

where $\gamma_1 = 5\nu_{7a}+5\nu_{7b}+5\nu_{7c}-2\nu_{7d}$, $\gamma_2 = \nu_{7a}+\nu_{7b}+\nu_{7c}$, $\gamma_3 = 9\nu_{7a}+9\nu_{7b}+9\nu_{7c}+2\nu_{7d}$, $\gamma_4 = 6\nu_{7a}+6\nu_{7b}+6\nu_{7c}-\nu_{7d}$, $\gamma_5 = 8\nu_{7a}+8\nu_{7b}+8\nu_{7c}+\nu_{7d}$, $\gamma_6 = 4\nu_{7a}+4\nu_{7b}+4\nu_{7c}-3\nu_{7d}$, $\gamma_7 = 3\nu_{7a}+3\nu_{7b}+3\nu_{7c}-4\nu_{7d}$, $\gamma_8 = -8\nu_{7a}-\nu_{7b}+13\nu_{7c}-\nu_{7d}$, $\gamma_9 = -\nu_{7a}+13\nu_{7b}-8\nu_{7c}-\nu_{7d}$, $\gamma_{10} = 13\nu_{7a}-8\nu_{7b}-\nu_{7c}-\nu_{7d}$, $\gamma_{11} = 36\nu_{7a}-27\nu_{7b}-6\nu_{7c}+\nu_{7d}$, $\gamma_{12} = -6\nu_{7a}+36\nu_{7b}-27\nu_{7c}+\nu_{7d}$, $\gamma_{13} = -27\nu_{7a}-6\nu_{7b}+36\nu_{7c}+\nu_{7d}$, $\gamma_{14} = -22\nu_{7a}-22\nu_{7b}-22\nu_{7c}+6\nu_{7d}$, $\gamma_{15} = 34\nu_{7a}-\nu_{7b}-22\nu_{7c}-\nu_{7d}$, $\gamma_{16} = -22\nu_{7a}+34\nu_{7b}-\nu_{7c}-\nu_{7d}$, $\gamma_{17} = -\nu_{7a}-22\nu_{7b}+34\nu_{7c}-\nu_{7d}$, $\gamma_{18} = -20\nu_{7a}+8\nu_{7b}+15\nu_{7c}+\nu_{7d}$, $\gamma_{19} = 15\nu_{7a}-20\nu_{7b}+8\nu_{7c}+\nu_{7d}$, $\gamma_{20} = 8\nu_{7a}+15\nu_{7b}-20\nu_{7c}+\nu_{7d}$, $\gamma_{21} = -34\nu_{7a}-34\nu_{7b}-34\nu_{7c}-6\nu_{7d}$, $\gamma_{22} = -20\nu_{7a}-13\nu_{7b}+50\nu_{7c}+\nu_{7d}$, $\gamma_{23} = 50\nu_{7a}-20\nu_{7b}-13\nu_{7c}+\nu_{7d}$, $\gamma_{24} = -13\nu_{7a}+50\nu_{7b}-20\nu_{7c}+\nu_{7d}$, $\gamma_{25} = -35\nu_{7a}+28\nu_{7b}+7\nu_{7c}$, $\gamma_{26} =$

$7\nu_{7a} - 35\nu_{7b} + 28\nu_{7c}$, $\gamma_{27} = 28\nu_{7a} + 7\nu_{7b} - 35\nu_{7c}$, $\gamma_{28} = 35\nu_{7a} - 14\nu_{7b} - 14\nu_{7c}$, $\gamma_{29} = -14\nu_{7a} + 35\nu_{7b} - 14\nu_{7c}$ and $\gamma_{30} = -14\nu_{7a} - 14\nu_{7b} + 35\nu_{7c}$.

It follows that the only possible integer solutions for $(\nu_{7a}, \nu_{7b}, \nu_{7c}, \nu_{7d})$ are $(1, 0, 0, 0)$, $(0, 0, 0, 1)$, $(0, 0, 1, 0)$ and $(0, 1, 0, 0)$. Therefore, u is rationally conjugated to some element $g \in G$ by Proposition 4.

Case (iv). Let $u \in V(\mathbb{Z}G)$ where $|u| = 13$. Using Propositions 2 and 3,

$$\nu_{13a} + \nu_{13b} + \nu_{13c} = 1.$$

Applying Proposition 5, we obtain:

$$\mu_1(u, \chi_9, 2) = \tfrac{1}{13}(\gamma_1 + 160) \geq 0; \quad \mu_1(u, \chi_6, 2) = \tfrac{1}{13}(-\gamma_1 + 48) \geq 0;$$
$$\mu_2(u, \chi_6, 2) = \tfrac{1}{13}(\gamma_2 + 48) \geq 0; \quad \mu_2(u, \chi_9, 2) = \tfrac{1}{13}(-\gamma_2 + 160) \geq 0;$$
$$\mu_1(u, \chi_2, 2) = \tfrac{1}{13}(\gamma_3 + 8) \geq 0; \quad \mu_2(u, \chi_2, 2) = \tfrac{1}{13}(\gamma_3 + 8) \geq 0;$$
$$\mu_4(u, \chi_2, 2) = \tfrac{1}{13}(\gamma_4 + 8) \geq 0$$

where $\gamma_1 = 9\nu_{13a} - 4\nu_{13b} - 4\nu_{13c}$, $\gamma_2 = 4\nu_{13a} - 9\nu_{13b} + 4\nu_{13c}$, $\gamma_3 = 5\nu_{13a} - 8\nu_{13b} + 5\nu_{13c}$, $\gamma_4 = 5\nu_{13a} + 5\nu_{13b} - 8\nu_{13c}$ and $\gamma_5 = -8\nu_{13a} + 5\nu_{13b} + 5\nu_{13c}$. Clearly $\gamma_1 \in \{9 + 13k \mid -13 \leq k \leq 3\}$ and $\gamma_2 \in \{4 + 13k \mid -4 \leq k \leq 12\}$. It follows that the only possible integer solutions for $(\nu_{13a}, \nu_{13b}, \nu_{13c})$ are listed in part (v) of Theorem 1.

Case (v). Let $u \in V(\mathbb{Z}G)$ where $|u| = 26$. Using Propositions 2 and 3,

$$\nu_{2a} + \nu_{2b} + \nu_{13a} + \nu_{13b} + \nu_{13c} = 1.$$

Let $\gamma_1 = 3\nu_{2a} - \nu_{2b}$, $\gamma_2 = 5\nu_{2a} - \nu_{2b}$, $\gamma_3 = -28\nu_{2a} - 4\nu_{2b} + \nu_{13a} + \nu_{13b} + \nu_{13c}$, $\gamma_4 = 63\nu_{2a} + 15\nu_{2b} - 4\nu_{13a} - 4\nu_{13b} + 9\nu_{13c}$ and $\gamma_5 = 63\nu_{2a} + 15\nu_{2b} + 9\nu_{13a} - 4\nu_{13b} - 4\nu_{13c}$. We shall now separately consider the following cases involving $\chi(u^n)$ for $n \in \{2, 13\}$:

- $\chi(u^{13}) = m_1\chi(2a) + m_2\chi(2b)$ and $\chi(u^2) = m_3\chi(13a) + m_4\chi(13b) + m_5\chi(13c)$ where

$$\begin{aligned}(m_1, m_2, m_3, m_4, m_5) \in \{&(1, 0, 1, 0, 0), (1, 0, 0, 1, 0), (1, 0, 0, 0, 1), (1, 0, 1, -1, 1),\\ &(1, 0, 1, 1, -1), (1, 0, -1, 1, 1), (2, -1, 1, 0, 0),\\ &(2, -1, 0, 1, 0), (2, -1, 0, 0, 1), (2, -1, 1, -1, 1),\\ &(2, -1, 1, 1, -1), (2, -1, -1, 1, 1), (-1, 2, 1, 0, 0),\\ &(-1, 2, 0, 1, 0), (-1, 2, 0, 0, 1), (-1, 2, 1, -1, 1),\\ &(-1, 2, 1, 1, -1), (-1, 2, -1, 1, 1), (-2, 3, 1, 0, 0),\\ &(-2, 3, 0, 1, 0), (-2, 3, 0, 0, 1), (-2, 3, 1, -1, 1),\\ &(-2, 3, 1, 1, -1), (-2, 3, -1, 1, 1)\}.\end{aligned}$$

Applying Proposition 5, we obtain:

$$\mu_{13}(u, \chi_2, *) = \tfrac{1}{26}(24\gamma_1 + 8) \geq 0; \quad \mu_0(u, \chi_2, *) = \tfrac{1}{26}(-24\gamma_1 + 44) \geq 0.$$

It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{13a}, \nu_{13b}, \nu_{13c})$ in all cases.

- $\chi(u^{13}) = m_1\chi(2a) + m_2\chi(2b)$ and $\chi(u^2) = m_3\chi(13a) + m_4\chi(13b) + m_5\chi(13c)$ where

$$\begin{aligned}(m_1, m_2, m_3, m_4, m_5) \in \{&(0, 1, 1, 0, 0), (0, 1, 0, 1, 0), (0, 1, 0, 0, 1),\\ &(0, 1, 1, -1, 1), (0, 1, 1, 1, -1), (0, 1, -1, 1, 1)\}.\end{aligned}$$

Applying Proposition 5, we obtain:

$$\mu_{13}(u,\chi_2,*)=\tfrac{1}{26}(24\gamma_1+24)\geq 0;\quad \mu_0(u,\chi_2,*)=\tfrac{1}{26}(-24\gamma_1+28)\geq 0;$$
$$\mu_0(u,\chi_3,*)=\tfrac{1}{26}(48\gamma_2+48)\geq 0;\quad \mu_{13}(u,\chi_3,*)=\tfrac{1}{26}(-48\gamma_2+56)\geq 0;$$
$$\mu_1(u,\chi_4,*)=\tfrac{1}{26}(\gamma_3+199)\geq 0;\quad \mu_8(u,\chi_{31},*)=\tfrac{1}{26}(\gamma_4+k_1)\geq 0;$$
$$\mu_1(u,\chi_{31},*)=\tfrac{1}{26}(-\gamma_4+k_2)\geq 0;\quad \mu_2(u,\chi_{31},*)=\tfrac{1}{26}(\gamma_5+k_3)\geq 0;$$
$$\mu_3(u,\chi_{31},*)=\tfrac{1}{26}(-\gamma_5+k_4)\geq 0$$

where $(m_1,m_2,m_3,m_4,m_5)=(0,1,1,0,0)$ when $(k_1,k_2,k_3,k_4)=(3993,3950,3980,3963)$, $(m_1,m_2,m_3,m_4,m_5)=(0,1,0,1,0)$ when $(k_1,k_2,k_3,k_4)=(3980,3963,3993,3950)$, $(m_1,m_2,m_3,m_4,m_5)=(0,1,0,0,1)$ when $(k_1,k_2,k_3,k_4)=(3980,3950,3980,3950)$, $(m_1,m_2,m_3,m_4,m_5)=(0,1,1,-1,1)$ when $(k_1,k_2,k_3,k_4)=(3993,3937,3967,3963)$, $(m_1,m_2,m_3,m_4,m_5)=(0,1,1,1,-1)$ when $(k_1,k_2,k_3,k_4)=(3993,3963,3993,3963)$ and $(m_1,m_2,m_3,m_4,m_5)=(0,1,-1,1,1)$ when $(k_1,k_2,k_3,k_4)=(3967,3963,3993,3937)$. It follows that there are no possible integer solutions for $(\nu_{2a},\nu_{2b},\nu_{13a},\nu_{13b},\nu_{13c})$ in all cases.

Case (vi). Let $u\in V(\mathbb{Z}G)$ where $|u|=39$. Using Propositions 2 and 3,

$$\nu_{3a}+\nu_{3b}+\nu_{13a}+\nu_{13b}+\nu_{13c}=1.$$

Consider the cases $\chi(u^{13})=m_1\chi(3a)+m_2\chi(3b)$ and $\chi(u^3)=m_3\chi(13a)+m_4\chi(13b)+m_5\chi(13c)$ where

$$\begin{aligned}(m_1,m_2,m_3,m_4,m_5)\in\{&(1,0,1,0,0),(1,0,0,1,0),(1,0,0,0,1),(0,1,1,0,0),\\&(0,1,0,1,0),(0,1,0,0,1),(1,0,1,-1,1),(0,1,1,-1,1),\\&(1,0,1,1,-1),(0,1,1,1,-1),(1,0,-1,1,1),(0,1,-1,1,1),\\&(3,-2,1,0,0),(3,-2,0,1,),(3,-2,0,0,1),(3,-2,1,-1,1),\\&(3,-2,1,1,-1),(3,-2,-1,1,1),(2,-1,1,0,0),\\&(2,-1,0,1,0),(2,-1,0,0,1),(2,-1,1,-1,1),\\&(2,-1,1,1,-1),(2,-1,-1,1,1),(-1,2,1,0,0),\\&(-1,2,0,1,0),(-1,2,0,0,1),(-1,2,1,-1,1),\\&(-1,2,1,1,-1),(-1,2,-1,1,1)\}.\end{aligned}$$

Applying Proposition 5, we obtain:

$$\mu_{13}(u,\chi_2,*)=\tfrac{1}{39}(12\gamma+27)\geq 0;\quad \mu_0(u,\chi_2,*)=\tfrac{1}{39}(-24\gamma+24)\geq 0;$$
$$\mu_1(u,\chi_2,*)=\tfrac{1}{39}(-\gamma+27)\geq 0$$

where $\gamma=\nu_{3a}+\nu_{3b}$. It follows that there are no possible integer solutions for $(\nu_{3a},\nu_{3b},\nu_{13a},\nu_{13b},\nu_{13c})$.

Case (vii). Let $u\in V(\mathbb{Z}G)$ where $|u|=91$. Using Propositions 2 and 3,

$$\nu_{7a}+\nu_{7b}+\nu_{7c}+\nu_{7d}+\nu_{13a}+\nu_{13b}+\nu_{13c}=1.$$

Let $\gamma=5\nu_{7a}+5\nu_{7b}+5\nu_{7c}-2\nu_{7d}$. We shall now separately consider the following cases involving $\chi(u^n)$ for $n\in\{7,13\}$:

- $\chi(u^{13})=\chi(7k)$ and $\chi(u^7)=m_1\chi(13a)+m_2\chi(13b)+m_3\chi(13c)$ where $k\in\{a,b,c\}$ and $(m_1,m_2,m_3)\in\{(1,0,0),(0,1,0),(0,0,1),(1,-1,1),(1,1,-1),(-1,1,1)\}$.

Applying Proposition 5, we obtain:

$$\mu_0(u, \chi_2, *) = \frac{1}{91}(72\gamma + 56) \geq 0; \quad \mu_{13}(u, \chi_2, *) = \frac{1}{91}(-12\gamma + 21) \geq 0.$$

It follows that there are no possible integer values for $(\nu_{7a}, \nu_{7b}, \nu_{7c}, \nu_{7d}, \nu_{13a}, \nu_{13b}, \nu_{13c})$.

- $\chi(u^{13}) = \chi(7d)$ and $\chi(u^7) = m_1\chi(13a) + m_2\chi(13b) + m_3\chi(13c)$ where

$$(m_1, m_2, m_3) \in \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, -1, 1), (1, 1, -1), (-1, 1, 1)\}.$$

Applying Proposition 5, we obtain:

$$\mu_0(u, \chi_2, *) = \frac{1}{91}(72\gamma + 14) \geq 0; \quad \mu_7(u, \chi_2, *) = \frac{1}{91}(-6\gamma + 14) \geq 0.$$

It follows that there are no possible integer values for $(\nu_{7a}, \nu_{7b}, \nu_{7c}, \nu_{7d}, \nu_{13a}, \nu_{13b}, \nu_{13c})$.

We shall now consider the prime graph of $G = {}^3D_4(2^3)$. G contains elements of order 6, 14 and 21. Therefore [2, 3], [2, 7] and [3, 7] are adjacent in $\pi(G)$ and consequently adjacent in $\pi(V(\mathbb{Z}G))$. Clearly $\pi(G) = \pi(V(\mathbb{Z}G))$, since there are no torsion units of order 26, 39 and 91 in $V(\mathbb{Z}G)$. This completes the proof.

3 Proof of Theorem 2

Let $G = {}^2F_4(2)'$. Clearly $|G| = 17971200 = 2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$ and $exp(G) = 3120 = 2^4 \cdot 3 \cdot 5 \cdot 13$. Initially, for any torsion unit of $V(\mathbb{Z}G)$ of order k:

$$\begin{aligned}&\nu_{2a} + \nu_{2b} + \nu_{3a} + \nu_{4a} + \nu_{4b} + \nu_{4c} + \nu_{5a} + \nu_{6a} + \nu_{8a} + \nu_{8b} + \nu_{8c} + \nu_{8d}\\ &\quad + \nu_{10a} + \nu_{12a} + \nu_{12b} + \nu_{13a} + \nu_{13b} + \nu_{16a} + \nu_{16b} + \nu_{16c} + \nu_{16d} = 1\end{aligned}$$

For the purpose of this paper, we shall consider torsion units of $V(\mathbb{Z}G)$ of order 2, 3, 5, 10, 15, 26, 39 and 65. We shall now consider each case separately.

Case (i). Let $u \in V(\mathbb{Z}G)$ where $|u| = 2$. Using Propositions 2 and 3,

$$\nu_{2a} + \nu_{2b} = 1.$$

Applying Proposition 5, we obtain the following system of inequalities:

$$\mu_0(u, \chi_2, *) = \frac{1}{2}(-2\gamma + 26) \geq 0; \quad \mu_1(u, \chi_2, *) = \frac{1}{2}(2\gamma + 26) \geq 0$$

where $\gamma = 3\nu_{2a} - \nu_{2b}$. It follows that the only possible integer solutions for (ν_{2a}, ν_{2b}) are listed in part (iii) of Theorem 2.

Case (ii). Let $u \in V(\mathbb{Z}G)$ where $|u| = 3$. By Proposition 2, $\nu_{kx} = 0$ for all

$$kx \in \{2a, 2b, 4a, 4b, 4c, 5a, 6a, 8a, 8b, 8c, 8d, 10a, 12a, 12b, 13a, 13b, 16a, 16b, 16c, 16d\}.$$

Therefore, u is rationally conjugated to some element $g \in G$ by Proposition 4.

Case (iii). Let $u \in V(\mathbb{Z}G)$ where $|u| = 5$. By Proposition 2, $\nu_{kx} = 0$ for all

$$kx \in \{2a, 2b, 3a, 4a, 4b, 4c, 6a, 8a, 8b, 8c, 8d, 10a, 12a, 12b, 13a, 13b, 16a, 16b, 16c, 16d\}.$$

Therefore, u is rationally conjugated to some element $g \in G$ by Proposition 4.

Case (iv). Let $u \in V(\mathbb{Z}G)$ where $|u| = 10$. Using Propositions 2 and 3,

$$\nu_{2a} + \nu_{2b} + \nu_{5a} + \nu_{10a} = 1.$$

Let $\gamma_1 = 6\nu_{2a} - 2\nu_{2b} - \nu_{5a} + \nu_{10a}$, $\gamma_2 = 14\nu_{2a} - 2\nu_{2b} + 3\nu_{5a} - \nu_{10a}$, $\gamma_3 = 5\nu_{2a} + \nu_{2b}$, $\gamma_4 = 5\nu_{2a} - 11\nu_{2b}$ and $\gamma_5 = 32\nu_{2a} + 15\nu_{2b} + \nu_{5a} + \nu_{10a}$. We shall now separately consider the following cases involving $\chi(u^n)$ for $n \in \{2, 5\}$:

- $\chi(u^5) = m_1\chi(2a) + m_2\chi(2b)$ and $\chi(u^2) = \chi(5a)$ where $(m_1, m_2) \in \{(1,0), (0,1), (-1,2)\}$. Applying Proposition 5, we obtain:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{10}(-4\gamma_1 + k_1) \geq 0; \quad \mu_1(u, \chi_2, *) = \tfrac{1}{10}(-\gamma_1 + k_2) \geq 0;$$
$$\mu_5(u, \chi_2, *) = \tfrac{1}{10}(4\gamma_1 + k_3) \geq 0; \quad \mu_0(u, \chi_6, *) = \tfrac{1}{10}(4\gamma_2 + k_4) \geq 0;$$
$$\mu_1(u, \chi_6, *) = \tfrac{1}{10}(\gamma_2 + k_5) \geq 0; \quad \mu_5(u, \chi_6, *) = \tfrac{1}{10}(-4\gamma_2 + k_6) \geq 0$$
$$\mu_0(u, \chi_7, *) = \tfrac{1}{10}(-16\gamma_3 + k_7) \geq 0; \quad \mu_5(u, \chi_7, *) = \tfrac{1}{10}(16\gamma_3 + k_8) \geq 0;$$
$$\mu_0(u, \chi_9, *) = \tfrac{1}{10}(4\gamma_5 + k_9) \geq 0; \quad \mu_5(u, \chi_9, *) = \tfrac{1}{10}(-4\gamma_5 + k_{10}) \geq 0$$

where the values for k_i's and corresponding (m_1, m_2) values are as follows:

(m_1, m_2)	$(k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9, k_{10})$
(1, 0)	(24, 31, 36, 104, 61, 76, 280, 320, 386, 324)
(0, 1)	(32, 23, 28, 88, 77, 92, 296, 304, 370, 340)
(−1, 2)	(40, 15, 20, 72, 93, 108, 312, 288, 354, 356)

It follows that the only possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{13a})$ are $(0, 0, 5, -4)$, $(0, 0, 0, 1)$, $(0, -4, 5, 0)$, $(1, 1, 0, -1)$, $(-1, 1, 0, 1)$, $(-1, -3, 0, 5)$, $(-1, -3, 5, 0)$, $(2, 2, 0, -3)$ and $(0, 2, 0, -1)$ in all cases.

- $\chi(u^5) = 3\chi(2a) - 2\chi(2b)$ and $\chi(u^2) = \chi(5a)$. Applying Proposition 5, we obtain:

$$\mu_2(u, \chi_2, *) = \tfrac{1}{10}(\gamma_1 + 3) \geq 0; \quad \mu_0(u, \chi_2, *) = \tfrac{1}{10}(-4\gamma_1 + 8) \geq 0;$$
$$\mu_1(u, \chi_6, *) = \tfrac{1}{10}(\gamma_2 + 29) \geq 0; \quad \mu_5(u, \chi_6, *) = \tfrac{1}{10}(-4\gamma_2 + 44) \geq 0;$$
$$\mu_5(u, \chi_7, *) = \tfrac{1}{10}(16\gamma_3 + 352) \geq 0; \quad \mu_0(u, \chi_7, *) = \tfrac{1}{10}(-16\gamma_3 + 248) \geq 0;$$
$$\mu_0(u, \chi_9, *) = \tfrac{1}{10}(4\gamma_5 + 418) \geq 0.$$

Clearly $\gamma_1 = -3$, $\gamma_2 \in \{-29, -19, -9, 1, 11\}$ and $\gamma_3 \in \{3 + 5k \mid -5 \leq k \leq 2\}$. It follows that the only possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{13a})$ are $(-2, -2, 0, 5)$ and $(1, 3, 0, -3)$.

- $\chi(u^5) = 2\chi(2a) - \chi(2b)$ and $\chi(u^2) = \chi(5a)$. Applying Proposition 5, we obtain:

$$\mu_2(u, \chi_2, *) = \tfrac{1}{10}(\gamma_1 + 11) \geq 0; \quad \mu_0(u, \chi_2, *) = \tfrac{1}{10}(-4\gamma_1 + 16) \geq 0;$$
$$\mu_0(u, \chi_6, *) = \tfrac{1}{10}(4\gamma_2 + 120) \geq 0; \quad \mu_5(u, \chi_6, *) = \tfrac{1}{10}(-4\gamma_2 + 60) \geq 0;$$
$$\mu_1(u, \chi_6, *) = \tfrac{1}{10}(\gamma_2 + 45) \geq 0; \quad \mu_5(u, \chi_7, *) = \tfrac{1}{10}(16\gamma_3 + 336) \geq 0;$$
$$\mu_0(u, \chi_7, *) = \tfrac{1}{10}(-16\gamma_3 + 264) \geq 0; \quad \mu_0(u, \chi_8, *) = \tfrac{1}{10}(4\gamma_4 + 346) \geq 0;$$
$$\mu_0(u, \chi_9, *) = \tfrac{1}{10}(4\gamma_5 + 402) \geq 0.$$

Clearly $\gamma_1 \in \{-1, -11\}$, $\gamma_2 \in \{-25, -15, -5, 5, 15\}$ and $\gamma_3 \in \{4 + 5k \mid -5 \leq k \leq 2\}$. It follows that the only possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{13a})$ are $(-1, -1, 0, 3)$, $(-1, -1, 5, -2)$, $(0, 4, -5, 2)$ and $(0, 4, 0, -3)$.

- $\chi(u^5) = -2\chi(2a) + 3\chi(2b)$ and $\chi(u^2) = \chi(5a)$. Applying Proposition 5, we obtain:

$$\mu_5(u, \chi_2, *) = \tfrac{1}{10}(4\gamma_1 + 12) \geq 0; \quad \mu_1(u, \chi_2, *) = \tfrac{1}{10}(-\gamma_1 + 7) \geq 0;$$
$$\mu_0(u, \chi_6, *) = \tfrac{1}{10}(4\gamma_2 + 56) \geq 0; \quad \mu_5(u, \chi_6, *) = \tfrac{1}{10}(-4\gamma_2 + 124) \geq 0;$$
$$\mu_1(u, \chi_6, *) = \tfrac{1}{10}(\gamma_2 + 109) \geq 0; \quad \mu_5(u, \chi_7, *) = \tfrac{1}{10}(16\gamma_3 + 272) \geq 0;$$
$$\mu_0(u, \chi_7, *) = \tfrac{1}{10}(-16\gamma_3 + 328) \geq 0; \quad \mu_0(u, \chi_8, *) = \tfrac{1}{10}(\gamma_4 + 282) \geq 0;$$
$$\mu_0(u, \chi_9, *) = \tfrac{1}{10}(\gamma_5 + 338) \geq 0.$$

Clearly $\gamma_1 \in \{-3, 7\}$ $\gamma_2 \in \{-9, 1, 11, 21, 3\}$ and $\gamma_3 \in \{3 + 5k \mid -4 \leq k \leq\}$. It follows that the only possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{13a})$ are $(1, 3, 0, -3)$, $(0, -2, 5, -2)$, $(1, 3, -5, 2)$ and $(0, -2, 0, 3)$.

- $\chi(u^5) = -3\chi(2a) + 4\chi(2b)$ and $\chi(u^2)$. Applying Proposition 5, we obtain:

$$\mu_5(u, \chi_2, *) = \tfrac{1}{10}(4\gamma_1 + 4) \geq 0; \quad \mu_1(u, \chi_2, *) = \tfrac{1}{10}(-\gamma_1 - 1) \geq 0;$$
$$\mu_0(u, \chi_6, *) = \tfrac{1}{10}(4\gamma_2 + 40) \geq 0; \quad \mu_2(u, \chi_6, *) = \tfrac{1}{10}(-\gamma_2 + 25) \geq 0;$$
$$\mu_5(u, \chi_7, *) = \tfrac{1}{10}(16\gamma_3 + 256) \geq 0; \quad \mu_0(u, \chi_7, *) = \tfrac{1}{10}(-16\gamma_3 + 344) \geq 0;$$
$$\mu_0(u, \chi_9, *) = \tfrac{1}{10}(4\gamma_5 + 322) \geq 0; \quad \mu_5(u, \chi_9, *) = \tfrac{1}{10}(-4\gamma_5 + 388) \geq 0.$$

Clearly $\gamma_1 = -1$, $\gamma_2 \in \{-5, 5, 15, 25\}$ and $\gamma_3 \in \{4 + 5k \mid -4 \leq k \leq 3\}$. It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{13a})$ in this case.

Case (v). Let $u \in V(\mathbb{Z}G)$ where $|u| = 13$. Using Propositions 2 and 3,

$$\nu_{13a} + \nu_{13b} = 1.$$

Applying Proposition 5, we obtain the following system of inequalities:

$$\mu_1(u, \chi_7, 5) = \tfrac{1}{13}(\gamma_1 + 109) \geq 0; \quad \mu_2(u, \chi_{15}, 5) = \tfrac{1}{13}(-\gamma_1 + 593) \geq 0;$$
$$\mu_2(u, \chi_7, 5) = \tfrac{1}{13}(\gamma_2 + 109) \geq 0$$

where $\gamma_1 = 8\nu_{13a} - 5\nu_{13b}$ and $\gamma_2 = -5\nu_{13a} + 8\nu_{13b}$. It follows that the only possible integer solutions for (ν_{3a}, ν_{3b}) are listed in part (v) of Theorem 2.

Case (vi). Let $u \in V(\mathbb{Z}G)$ where $|u| = 15$. Using Propositions 2 and 3,

$$\nu_{3a} + \nu_{5a} = 1.$$

Applying Proposition 5, we obtain the following system of inequalities:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{15}(-8\gamma + 28) \geq 0; \quad \mu_5(u, \chi_2, *) = \tfrac{1}{15}(4\gamma + 31) \geq 0$$

where $\gamma = \nu_{3a} - \nu_{5a}$. Clearly $\gamma = -4$ and there are no possible integer solutions for (ν_{3a}, ν_{5a}).

Case (vii). Let $u \in V(\mathbb{Z}G)$ where $|u| = 26$. Using Propositions 2 and 3,

$$\nu_{2a} + \nu_{2b} + \nu_{13a} + \nu_{13b} = 1.$$

Let $\gamma_1 = 3\nu_{2a} - \nu_{2b}$, $\gamma_2 = 7\nu_{2a} - \nu_{2b}$, $\gamma_3 = -20\nu_{2a} - 4\nu_{2b} + \nu_{13a} + \nu_{13b}$ and $\gamma_4 = 6\nu_{13a} - 7\nu_{13b}$. We shall now separately consider the following cases involving $\chi(u^n)$ for $n \in \{2, 13\}$:

- $\chi(u^{13}) = m_1\chi(2a) + m_2\chi(2b)$ and $\chi(u^2) = m_3\chi(5a) + m_4\chi(5b)$ where $(m_1, m_2, m_3, m_4) \in \{(1, 0, 1, 0), (1, 0, 0, 1), (1, 0, 9, -8), (1, 0, 8, -7), (1, 0, 7, -6), (1, 0, 6, -5), (1, 0, 5, -4), (1, 0, 4, -3), (1, 0, 3, -2), (1, 0, 2, -1), (1, 0, -1, 2), (1, 0, -2, 3), (1,$

0, −3, 4), (1, 0, −4, 5), (1, 0, −5, 6), (1, 0, −6, 7), (1, 0, −7, 8), (1, 0, −8, 9)}. Applying Proposition 5, we obtain:

$$\mu_{13}(u,\chi_2,*)=\tfrac{1}{26}(24\gamma_1+32)\geq 0;\quad \mu_0(u,\chi_2,*)=\tfrac{1}{26}(-24\gamma_1+20)\geq 0.$$

It follows that there are no possible integer values for $(\nu_{2a},\nu_{2b},\nu_{13a},\nu_{13b})$.

- $\chi(u^{13})=m_1\chi(2a)+m_2\chi(2b)$ and $\chi(u^2)=m_3\chi(5a)+m_4\chi(5b)$ where $(m_1,m_2,m_3,m_4)\in\{(2,-1,1,0),(2,-1,0,1),(2,-1,9,-8),(2,-1,8,-7),(2,-1,7,-6),(2,-1,6,-5),(2,-1,5,-4),(2,-1,4,-3),(2,-1,3,-2),(2,-1,2-1),(2,-1,-1,2),(2,-1,-2,3),(2,-1,-3,4),(2,-1,-4,5),(2,-1,-5,6),(2,-1,-6,7),(2,-1,-7,8),(2,-1,-8,9)\}$. Applying Proposition 5, we obtain:

$$\mu_{13}(u,\chi_2,*)=\tfrac{1}{26}(24\gamma_1+16)\geq 0;\quad \mu_0(u,\chi_2,*)=\tfrac{1}{26}(-24\gamma_1+36)\geq 0.$$

It follows that there are no possible integer values for $(\nu_{2a},\nu_{2b},\nu_{13a},\nu_{13b})$.

- $\chi(u^{13})=m_1\chi(2a)+m_2\chi(2b)$ and $\chi(u^2)=m_3\chi(5a)+m_4\chi(5b)$ where $(m_1,m_2,m_3,m_4)\in\{(-2,3,1,0),(-2,3,0,1),(-2,3,9,-8),(-2,3,8,-7),(-2,3,7,-6),(-2,3,6,-5),(-2,3,5,-4),(-2,3,4,-3),(-2,3,3,-2),(-2,3,2,-1),(-2,3,-1,2),(-2,3,-2,3),(-2,3,-3,4),(-2,3,-4,5),(-2,3,-5,6),(-2,3,-6,7),(-2,3,-7,8),(-2,3,-8,9)\}$. Applying Proposition 5, we obtain:

$$\mu_{13}(u,\chi_2,*)=\tfrac{1}{26}(24\gamma_1+8)\geq 0;\quad \mu_0(u,\chi_2,*)=\tfrac{1}{26}(-24\gamma_1+44)\geq 0.$$

It follows that there are no possible integer values for $(\nu_{2a},\nu_{2b},\nu_{13a},\nu_{13b})$.

- $\chi(u^{13})=m_1\chi(2a)+m_2\chi(2b)$ and $\chi(u^2)=m_3\chi(5a)+m_4\chi(5b)$ where $(m_1,m_2,m_3,m_4)\in\{(0,1,9,-8),(0,1,-8,9)\}$. Applying Proposition 5, we obtain:

$$\mu_{13}(u,\chi_2,*)=\tfrac{1}{26}(24\gamma_1+24)\geq 0;\quad \mu_0(u,\chi_2,*)=\tfrac{1}{26}(-24\gamma_1+28)\geq 0;$$
$$\mu_0(u,\chi_6,*)=\tfrac{1}{26}(24\gamma_2+76)\geq 0;\quad \mu_{13}(u,\chi_6,*)=\tfrac{1}{26}(-24\gamma_2+80)\geq 0;$$
$$\mu_1(u,\chi_7,*)=\tfrac{1}{26}(\gamma_3+303)\geq 0.$$

Clearly $\gamma_1=\gamma_2=-1$. It follows that there are no possible integer values for $(\nu_{2a},\nu_{2b},\nu_{13a},\nu_{13b})$.

- $\chi(u^{13})=m_1\chi(2a)+m_2\chi(2b)$ and $\chi(u^2)=m_3\chi(5a)+m_4\chi(5b)$ where $(m_1,m_2,m_3,m_4)\in\{(3,-2,9,-8),(3,-2,-8,9)\}$. Applying Proposition 5, we obtain:

$$\mu_2(u,\chi_2,*)=\tfrac{1}{26}(2\gamma_1+4)\geq 0;\quad \mu_0(u,\chi_2,*)=\tfrac{1}{26}(-24\gamma_1+4)\geq 0;$$
$$\mu_0(u,\chi_6,*)=\tfrac{1}{26}(24\gamma_2+124)\geq 0;\quad \mu_{13}(u,\chi_6,*)=\tfrac{1}{26}(-24\gamma_2+32)\geq 0.$$

Clearly $\gamma_1=-2$ and $\gamma_2=-3$. It follows that there are no possible integer values for $(\nu_{2a},\nu_{2b},\nu_{13a},\nu_{13b})$.

- $\chi(u^{13})=m_1\chi(2a)+m_2\chi(2b)$ and $\chi(u^2)=m_3\chi(5a)+m_4\chi(5b)$ where $(m_1,m_2,m_3,m_4)\in\{(-3,4,9,-8),(-3,4,-8,9)\}$. Applying Proposition 5, we obtain:

$$\mu_{13}(u,\chi_2,*)=\tfrac{1}{26}(24\gamma_1)\geq 0;\quad \mu_1(u,\chi_2,*)=\tfrac{1}{26}(-2\gamma_1)\geq 0;$$
$$\mu_0(u,\chi_6,*)=\tfrac{1}{26}(24\gamma_2+28)\geq 0;\quad \mu_{13}(u,\chi_6,*)=\tfrac{1}{26}(-24\gamma_2+128)\geq 0.$$

Clearly $\gamma_1=0$ and $\gamma_2=1$. It follows that there are no possible integer values for $(\nu_{2a},\nu_{2b},\nu_{13a},\nu_{13b})$.

- $\chi(u^{13}) = m_1\chi(2a) + m_2\chi(2b)$ and $\chi(u^2) = m_3\chi(5a) + m_4\chi(5b)$ where $(m_1, m_2, m_3, m_4) \in \{(0, 1, 1, 0), (0, 1, 0, 1), (0, 1, 8, -7), (0, 1, 7, -6), (0, 1, 6, -5), (0, 1, 5, -4), (0, 1, 4, -3), (0, 1, 3, -2), (0, 1, 2, -1), (0, 1, -1, 2), (0, 1, -2, 3), (0, 1, -3, 4), (0, 1, -4, 5), (0, 1, -5, 6), (0, 1, -6, 7), (0, 1, -7, 8)\}$. Applying Proposition 5, we obtain:

$$\mu_{13}(u, \chi_2, *) = \tfrac{1}{26}(24\gamma_1 + 24) \geq 0; \quad \mu_0(u, \chi_2, *) = \tfrac{1}{26}(-24\gamma_1 + 28) \geq 0;$$
$$\mu_0(u, \chi_6, *) = \tfrac{1}{26}(24\gamma_2 + 76) \geq 0; \quad \mu_{13}(u, \chi_6, *) = \tfrac{1}{26}(-24\gamma_2 + 80) \geq 0;$$
$$\mu_1(u, \chi_7, *) = \tfrac{1}{26}(\gamma_3 + 303) \geq 0; \quad \mu_4(u, \chi_{21}, *) = \tfrac{1}{26}(\gamma_4 + k) \geq 0;$$
$$\mu_1(u, \chi_{21}, *) = \tfrac{1}{26}(-\gamma_4 + k) \geq 0$$

where the corresponding k values to the above cases are:

(m_1, m_2, m_3, m_4)	k	(m_1, m_2, m_3, m_4)	k
(0,1,1,0)	2041	(0,1,0,1)	2054
(0, 1, 8, −7)	1950	(0, 1, 7, −6)	1963
(0, 1, 6, −5)	1976	(0, 1, 5, −4)	1989
(0, 1, 4, −3)	2002	(0, 1, 3, −2)	2015
(0, 1, 2, −1)	2028	(0, 1, −1, 2)	2067
(0, 1, −2, 3)	2080	(0, 1, −3, 4)	2093
(0, 1, −4, 5)	2106	(0, 1, −5, 6)	2119
(0, 1, −6, 7)	2132	(0, 1, −7, 8)	2145

It follows that there are no possible integer values for $(\nu_{2a}, \nu_{2b}, \nu_{13a}, \nu_{13b})$ in all cases.

- $\chi(u^{13}) = m_1\chi(2a) + m_2\chi(2b)$ and $\chi(u^2) = m_3\chi(5a) + m_4\chi(5b)$ where $(m_1, m_2, m_3, m_4) \in \{(3, -2, 1, 0), (3, -2, 0, 1), (3, -2, 8, -7), (3, -2, 7, -6), (3, -2, 6, -5), (3, -2, 5, -4), (3, -2, 4, -3), (3, -2, 3, -2), (3, -2, 2, -1), (3, -2, -1, 2), (3, -2, -2, 3), (3, -2, -3, 4), (3, -2, -4, 5), (3, -2, -5, 6), (3, -2, -6, 7), (3, -2, -7, 8)\}$. Applying Proposition 5, we obtain:

$$\mu_2(u, \chi_2, *) = \tfrac{1}{26}(2\gamma_1 + 4) \geq 0; \quad \mu_0(u, \chi_2, *) = \tfrac{1}{26}(-24\gamma_1 + 4) \geq 0;$$
$$\mu_0(u, \chi_6, *) = \tfrac{1}{26}(24\gamma_2 + 124) \geq 0; \quad \mu_{13}(u, \chi_6, *) = \tfrac{1}{26}(-24\gamma_2 + 32) \geq 0;$$
$$\mu_4(u, \chi_{21}, *) = \tfrac{1}{26}(\gamma_4 + k) \geq 0; \quad \mu_1(u, \chi_{21}, *) = \tfrac{1}{26}(-\gamma_4 + k) \geq 0.$$

where the corresponding k-values to the above cases are:

(m_1, m_2, m_3, m_4)	k	(m_1, m_2, m_3, m_4)	k
(3, −2, 1, 0)	2041	(3, −2, 0, 1)	2054
(3, −2, 8, −7)	1950	(3, −2, 7, −6)	1963
(3, −2, 6, −5)	1976	(3, −2, 5, −4)	1989
(3, −2, 4, −3)	2002	(3, −2, 3, −2)	2015
(3, −2, 2, −1)	2028	(3, −2, −1, 2)	2067
(3, −2, −2, 3)	2080	(3, −2, −3, 4)	2093
(3, −2, −4, 5)	2106	(3, −2, −5, 6)	2119
(3, −2, −6, 7)	2132	(3, −2, −7, 8)	2145

It follows that there are no possible integer values for $(\nu_{2a}, \nu_{2b}, \nu_{13a}, \nu_{13b})$ in all cases.

$\chi(u^{13}) = m_1\chi(2a) + m_2\chi(2b)$ and $\chi(u^2) = m_3\chi(5a) + m_4\chi(5b)$ where $(m_1, m_2, m_3, m_4) \in \{(-3, 4, 1, 0), (-3, 4, 0, 1), (-3, 4, 8, -7), (-3, 4, 7, -6), (-3, 4, 6, -5), (-3, 4, 5, -4), (-3, 4, 4, -3), (-3, 4, 3, -2), (-3, 4, 2, -1), (-3, 4, -1, 2), (-3, 4, -2, 3), (-3, 4, -3, 4), (-3, 4, -4, 5), (-3, 4, -5, 6), (-3, 4, -6, 7), (-3, 4, -7, 8)\}$. Applying Proposition 5, we obtain:

$$\mu_{13}(u, \chi_2, *) = \tfrac{1}{26}(24\gamma_1) \geq 0; \quad \mu_1(u, \chi_2, *) = \tfrac{1}{26}(-2\gamma_1) \geq 0;$$
$$\mu_0(u, \chi_6, *) = \tfrac{1}{26}(24\gamma_2 + 28) \geq 0; \quad \mu_{13}(u, \chi_6, *) = \tfrac{1}{26}(-24\gamma_2 + 128) \geq 0;$$
$$\mu_4(u, \chi_{21}, *) = \tfrac{1}{26}(\gamma_4 + k) \geq 0; \quad \mu_1(u, \chi_{21}, *) = \tfrac{1}{26}(-\gamma_4 + k) \geq 0.$$

where the corresponding k values to the above cases are:

(m_1, m_2, m_3, m_4)	k	(m_1, m_2, m_3, m_4)	k
$(-3, 4, 1, 0)$	2041	$(-3, 4, 0, 1)$	2054
$(-3, 4, 8, -7)$	1950	$(-3, 4, 7, -6)$	1963
$(-3, 4, 6, -5)$	1976	$(-3, 4, 5, -4)$	1989
$(-3, 4, 4, -3)$	2002	$(-3, 4, 3, -2)$	2015
$(-3, 4, 2, -1)$	2028	$(-3, 4, -1, 2)$	2067
$(-3, 4, -2, 3)$	2080	$(-3, 4, -3, 4)$	2093
$(-3, 4, -4, 5)$	2106	$(-3, 4, -5, 6)$	2119
$(-3, 4, -6, 7)$	2132	$(-3, 4, -7, 8)$	2145

It follows that there are no possible integer values for $(\nu_{2a}, \nu_{2b}, \nu_{13a}, \nu_{13b})$ in all cases. Case $(viii)$. Let $u \in V(\mathbb{Z}G)$ where $|u| = 39$. Using Propositions 2 and 3,

$$\nu_{3a} + \nu_{13a} + \nu_{13b} = 1.$$

Consider the cases $\chi(u^{13}) = \chi(3a)$ and $\chi(u^3) = m_1\chi(13a) + m_2\chi(13b)$ where

$$(m_1, m_2) \in \{(1, 0), (0, 1), (9, -8), (8, -7), (7, -6), (6, -5), (5, -4), (4, -3), (3, -2), (2, -1), (-1, 2), (-2, 3), (-3, 4), (-4, 5), (-5, 6), (-6, 7), (-7, 8), (-8, 9)\}.$$

Applying Proposition 5, we obtain:

$$\mu_{13}(u, \chi_2, *) = \tfrac{1}{39}(12\nu_{3a} + 27) \geq 0; \quad \mu_0(u, \chi_2, *) = \tfrac{1}{39}(-24\nu_{3a} + 24) \geq 0;$$
$$\mu_1(u, \chi_2, *) = \tfrac{1}{39}(-\nu_{3a} + 27) \geq 0.$$

It follows that there are no possible integer solutions for $(\nu_{3a}, \nu_{13a}, \nu_{13b})$.

Case (ix). Let $u \in V(\mathbb{Z}G)$ where $|u| = 65$. Using Propositions 2 and 3,

$$\nu_{5a} + \nu_{13a} + \nu_{13b} = 1.$$

Consider the cases $\chi(u^{13}) = \chi(5a)$ and $\chi(u^5) = m_1\chi(13a) + m_2\chi(13b)$ where

$$(m_1, m_2) \in \{(1, 0), (0, 1), (9, -8), (8, -7), (7, -6), (6, -5), (5, -4), (4, -3), (3, -2), (2, -1), (-1, 2), (-2, 3), (-3, 4), (-4, 5), (-5, 6), (-6, 7), (-7, 8), (-8, 9)\}.$$

Applying Proposition 5, we obtain:

$$\mu_0(u,\chi_2,*)=\tfrac{1}{65}(48\nu_{5a}+30)\geq 0;\quad \mu_{13}(u,\chi_2,*)=\tfrac{1}{65}(-12\nu_{5a}+25)\geq 0.$$

It follows that there are no possible integer solutions for $(\nu_{5a},\nu_{13a},\nu_{13b})$.
We shall now consider the prime graph of $G={}^2F_4(2)'$. G contains elements of order 6 and 10. Therefore [2, 3] and [2, 5] are adjacent in $\pi(G)$ and consequently adjacent in $\pi(V(\mathbb{Z}G))$. Clearly $\pi(G)=\pi(V(\mathbb{Z}G))$, since there are no torsion units of order 15, 26, 39 and 65 in $V(\mathbb{Z}G)$. This completes the proof.

4 Proof of Theorem 3

Let $G={}^2F_4(2)$. Clearly $|G|=35942400=2^{12}\cdot 3^3\cdot 5^2\cdot 13$ and $exp(G)=3120=2^4\cdot 3\cdot 5\cdot 13$. Initially, for any torsion unit of $V(\mathbb{Z}G)$ of order k:

$$\begin{aligned}&\nu_{2a}+\nu_{2b}+\nu_{3a}+\nu_{4a}+\nu_{4b}+\nu_{4c}+\nu_{5a}+\nu_{6a}+\nu_{8a}+\nu_{8b}+\nu_{8c}\\&+\nu_{10a}+\nu_{12a}+\nu_{13a}+\nu_{16a}+\nu_{16b}+\nu_{4d}+\nu_{4e}+\nu_{4f}+\nu_{4g}\\&+\nu_{8d}+\nu_{8e}+\nu_{12b}+\nu_{12c}+\nu_{16c}+\nu_{16d}+\nu_{20a}+\nu_{20b}=1\end{aligned}$$

For the purpose of this paper, we shall consider torsion units of $V(\mathbb{Z}G)$ of order 2, 3, 5, 10, 15, 26, 39 and 65. We shall now consider each case separately.

Case (i). Let $u\in V(\mathbb{Z}G)$ where $|u|=2$. Using Propositions 2 and 3,

$$\nu_{2a}+\nu_{2b}=1.$$

Applying Proposition 5, we obtain the following system of inequalities:

$$\mu_1(u,\chi_3,*)=\tfrac{1}{2}(4\gamma+52)\geq 0;\quad \mu_0(u,\chi_3,*)=\tfrac{1}{2}(-4\gamma+52)\geq 0$$

where $\gamma=3\nu_{2a}-\nu_{2b}$. Clearly $\gamma\in\{k\mid -13\leq k\leq 13\}$. It follows that the only possible integer solutions for (ν_{2a},ν_{2b}) are listed in part (iii) of Theorem 3.

Case (ii). Let $u\in V(\mathbb{Z}G)$ where $|u|=3$. By Proposition 2, $\nu_{kx}=0$ for all

$$\begin{aligned}kx\in\{&2a,2b,4a,4b,4c,5a,6a,8a,8b,8c,10a,12a,13a,16a,\\&16b,4d,4e,4f,4g,8d,8e,12b,12c,16c,16d,20a,20b\}.\end{aligned}$$

Therefore, u is rationally conjugated to some element $g\in G$ by Proposition 4.

Case (iii). Let $u\in V(\mathbb{Z}G)$ where $|u|=5$. By Proposition 2, $\nu_{kx}=0$ for all

$$\begin{aligned}kx\in\{&2a,2b,3a,4a,4b,4c,6a,8a,8b,8c,10a,12a,13a,16a,\\&16b,4d,4e,4f,4g,8d,8e,12b,12c,16c,16d,20a,20b\}.\end{aligned}$$

Therefore, u is rationally conjugated to some element $g\in G$ by Proposition 4.

Case (iv). Let $u\in V(\mathbb{Z}G)$ where $|u|=10$. Using Propositions 2 and 3,

$$\nu_{2a}+\nu_{2b}+\nu_{5a}+\nu_{10a}=1.$$

Let $\gamma_1=6\nu_{2a}-2\nu_{2b}-\nu_{5a}+\nu_{10a}$, $\gamma_2=14\nu_{2a}-2\nu_{2b}+3\nu_{5a}-\nu_{10a}$, $\gamma_3=5\nu_{2a}+\nu_{2b}$, $\gamma_4=5\nu_{2a}-11\nu_{2b}$ and $\gamma_5=31\nu_{2a}+15\nu_{2b}+\nu_{5a}+\nu_{10a}$. We shall now separately consider the following cases involving $\chi(u^n)$ for $n\in\{2,5\}$:

- $\chi(u^5) = m_1\chi(2a) + m_2\chi(2b)$ and $\chi(u^2) = \chi(5a)$ where $(m_1, m_2) \in \{(1, 0), (0, 1), (-1, 2)\}$. Applying Proposition 5, we obtain:

$$\mu_0(u, \chi_3, *) = \tfrac{1}{10}(-8\gamma_1 + k_1) \geq 0; \quad \mu_5(u, \chi_3, *) = \tfrac{1}{10}(8\gamma_1 + k_2) \geq 0;$$
$$\mu_0(u, \chi_8, *) = \tfrac{1}{10}(4\gamma_2 + k_3) \geq 0; \quad \mu_1(u, \chi_8, *) = \tfrac{1}{10}(\gamma_2 + k_4) \geq 0;$$
$$\mu_5(u, \chi_8, *) = \tfrac{1}{10}(-4\gamma_2 + k_5) \geq 0; \quad \mu_0(u, \chi_{10}, *) = \tfrac{1}{10}(-16\gamma_3 + k_6) \geq 0;$$
$$\mu_5(u, \chi_{10}, *) = \tfrac{1}{10}(16\gamma_3 + k_7) \geq 0; \quad \mu_0(u, \chi_{14}, *) = \tfrac{1}{10}(4\gamma_4 + k_8) \geq 0;$$
$$\mu_5(u, \chi_{14}, *) = \tfrac{1}{10}(-4\gamma_4 + k_9) \geq 0$$

where the values for k_i's and corresponding (m_1, m_2) values are as follows:

(m_1, m_2)	$(k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9)$
(1, 0)	(48, 72, 104, 61, 76, 280, 320, 386, 324)
(0, 1)	(64, 56, 88, 77, 92, 296, 304, 370, 340)
(−1, 2)	(80, 40, 72, 93, 108, 312, 288, 354, 356)

It follows that the only possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{13a})$ are $(0, 0, 5, -4)$, $(0, 0, 0, 1)$, $(-1, 1, 0, 1)$, $(1, 1, 0, -1)$, $(0, -4, 5, 0)$, $(0, 2, 0, -1)$, $(-1, -3, 5, 0)$, $(-1, -3, 0, 5)$ and $(2, 2, 0, -3)$ in all cases.

- $\chi(u^5) = 2\chi(2a) - \chi(2b)$ and $\chi(u^2) = \chi(5a)$. Applying Proposition 5, we obtain:

$$\mu_2(u, \chi_3, *) = \tfrac{1}{10}(2\gamma_1 + 22) \geq 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{10}(-8\gamma_1 + 32) \geq 0;$$
$$\mu_0(u, \chi_8, *) = \tfrac{1}{10}(4\gamma_2 + 120) \geq 0; \quad \mu_5(u, \chi_8, *) = \tfrac{1}{10}(-4\gamma_2 + 60) \geq 0;$$
$$\mu_1(u, \chi_8, *) = \tfrac{1}{10}(\gamma_2 + 45) \geq 0; \quad \mu_5(u, \chi_{10}, *) = \tfrac{1}{10}(16\gamma_3 + 336) \geq 0;$$
$$\mu_0(u, \chi_{10}, *) = \tfrac{1}{10}(-16\gamma_3 + 264) \geq 0; \quad \mu_0(u, \chi_{12}, *) = \tfrac{1}{10}(4\gamma_4 + 346) \geq 0;$$
$$\mu_0(u, \chi_{14}, *) = \tfrac{1}{10}(4\gamma_5 + 402) \geq 0.$$

Clearly $\gamma_1 \in \{-11, -6, -1, 4\}$, $\gamma_2 \in \{4+5k \mid -5 \leq k \leq 2\}$ and $\gamma_3 \in \{4+5k \mid -5 \leq k \leq 2\}$. It follows that the only possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{13a})$ are $(-1, -1, 0, 3)$, $(-1, -1, 5, -2)$, $(0, 4, -5, 2)$ and $(0, 4, 0, -3)$.

- $\chi(u^5) = -2\chi(2a) + 3\chi(2b)$ and $\chi(u^2) = \chi(5a)$. Applying Proposition 5, we obtain:

$$\mu_5(u, \chi_3, *) = \tfrac{1}{10}(8\gamma_1 + 24) \geq 0; \quad \mu_1(u, \chi_3, *) = \tfrac{1}{10}(-2\gamma_1 + 14) \geq 0;$$
$$\mu_0(u, \chi_8, *) = \tfrac{1}{10}(4\gamma_2 + 56) \geq 0; \quad \mu_5(u, \chi_8, *) = \tfrac{1}{10}(-4\gamma_2 + 124) \geq 0;$$
$$\mu_1(u, \chi_8, *) = \tfrac{1}{10}(\gamma_2 + 109) \geq 0; \quad \mu_5(u, \chi_{10}, *) = \tfrac{1}{10}(16\gamma_3 + 272) \geq 0;$$
$$\mu_0(u, \chi_{10}, *) = \tfrac{1}{10}(-16\gamma_3 + 328) \geq 0; \quad \mu_0(u, \chi_{12}, *) = \tfrac{1}{10}(4\gamma_4 + 282) \geq 0;$$
$$\mu_0(u, \chi_{14}, *) = \tfrac{1}{10}(4\gamma_5 + 338) \geq 0.$$

Clearly $\gamma_1 \in \{-3, 2, 7\}$, $\gamma_2 \in \{-9, 1, 11, 21, 31\}$ and $\gamma_3 \in \{3+5k \mid -4 \leq k \leq 3\}$. It follows that the only possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{13a})$ are $(0, -2, 0, 3)$, $(0, -2, 5, -2)$, $(1, 3, -5, 2)$ and $(1, 3, 0, -3)$.

- $\chi(u^5) = 3\chi(2a) - 2\chi(2b)$ and $\chi(u^2) = \chi(5a)$. Applying Proposition 5, we obtain:

$$\mu_2(u, \chi_3, *) = \tfrac{1}{10}(2\gamma_1 + 6) \geq 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{10}(-8\gamma_1 + 16) \geq 0;$$
$$\mu_1(u, \chi_8, *) = \tfrac{1}{10}(\gamma_2 + 29) \geq 0; \quad \mu_5(u, \chi_8, *) = \tfrac{1}{10}(-4\gamma_2 + 44) \geq 0;$$
$$\mu_5(u, \chi_{10}, *) = \tfrac{1}{10}(16\gamma_3 + 352) \geq 0; \quad \mu_0(u, \chi_{10}, *) = \tfrac{1}{10}(-16\gamma_3 + 248) \geq 0;$$
$$\mu_0(u, \chi_{14}, *) = \tfrac{1}{10}(4\gamma_5 + 418) \geq 0.$$

Clearly $\gamma_1 \in \{-3, 2\}$ $\gamma_2 \in \{-29, -19, -9, 1, 11\}$ and $\gamma_3 \in \{3 + 5k \mid -5 \leq k \leq 2\}$. It follows that the only possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{13a})$ are $(-2, -2, 0, 5)$ and $(1, 3, 0, -3)$.

- $\chi(u^5) = -3\chi(2a) + 4\chi(2b)$ and $\chi(u^2) = \chi(5a)$. Applying Proposition 5, we obtain:

$$\mu_5(u, \chi_3, *) = \tfrac{1}{10}(8\gamma_1 + 8) \geq 0; \quad \mu_1(u, \chi_3, *) = \tfrac{1}{10}(-2\gamma_1 - 2) \geq 0;$$
$$\mu_0(u, \chi_8, *) = \tfrac{1}{10}(4\gamma_2 + 40) \geq 0; \quad \mu_2(u, \chi_8, *) = \tfrac{1}{10}(-\gamma_2 + 25) \geq 0;$$
$$\mu_5(u, \chi_{10}, *) = \tfrac{1}{10}(16\gamma_3 + 256) \geq 0; \quad \mu_0(u, \chi_{10}, *) = \tfrac{1}{10}(-16\gamma_3 + 344) \geq 0;$$
$$\mu_0(u, \chi_{14}, *) = \tfrac{1}{10}(4\gamma_5 + 322) \geq 0; \quad \mu_5(u, \chi_{14}, *) = \tfrac{1}{10}(-4\gamma_5 + 388) \geq 0.$$

Clearly $\gamma_1 = -1$, $\gamma_2 \in \{-5, 5, 15, 25\}$ and $\gamma_3 \in \{4+5k \mid -4 \leq k \leq 3\}$. It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{13a})$ in this case.

Case (v). Let $u \in V(\mathbb{Z}G)$ where $|u| = 13$. By Proposition 2, $\nu_{kx} = 0$ for all

$$kx \in \{2a, 2b, 3a, 4a, 4b, 4c, 5a, 6a, 8a, 8b, 8c, 10a, 12a, 16a, 16b, 4d, 4e, 4f, 4g, 8d, 8e, 12b, 12c, 16c, 16d, 20a, 20b\}.$$

Therefore, u is rationally conjugated to some element $g \in G$ by Proposition 4.

Case (vi). Let $u \in V(\mathbb{Z}G)$ where $|u| = 15$. Using Propositions 2 and 3,

$$\nu_{3a} + \nu_{5a} = 1.$$

Applying Proposition 5, we obtain the following system of inequalities:

$$\mu_5(u, \chi_3, *) = \tfrac{1}{15}(8\gamma + 62) \geq 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{15}(-16\gamma + 56) \geq 0$$

where $\gamma = \nu_{3a} - \nu_{5a}$. Clearly $\gamma = -4$ and there are no possible integer solutions for (ν_{3a}, ν_{5a}).

Case (vii). Let $u \in V(\mathbb{Z}G)$ where $|u| = 26$. Using Propositions 2 and 3,

$$\nu_{2a} + \nu_{2b} + \nu_{13a} = 1.$$

Let $\gamma_1 = 3\nu_{2a} - \nu_{2b}$, $\gamma_2 = 7\nu_{2a} - \nu_{2b}$ and $\gamma_3 = -20\nu_{2a} - 4\nu_{2b} + \nu_{13a}$. We shall now separately consider the following cases involving $\chi(u^n)$ for $n \in \{2, 13\}$:

- $\chi(u^{13}) = m_1\chi(2a) + m_2\chi(2b)$ and $\chi(u^2) = \chi(13a)$ where $(m_1, m_2) \in \{(1, 0), (2, -1), (-1, 2), (-2, 3)\}$. Applying Proposition 5, we obtain:

$$\mu_{13}(u, \chi_3, *) = \tfrac{1}{26}(48\gamma_1 + k_1) \geq 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{26}(-48\gamma_1 + k_2) \geq 0$$

where $(k_1, k_2) = (64, 40)$ when $(m_1, m_2) = (1, 0)$, $(k_1, k_2) = (80, 24)$ when $(m_1, m_2) = (2, -1)$, $(k_1, k_2) = (32, 72)$ when $(m_1, m_2) = (-1, 2)$ and $(k_1, k_2) = (16, 80)$ when $(m_1, m_2) = (-2, 3)$. It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{13a})$ in all cases.

- $\chi(u^{13}) = -3\chi(2a)+4\chi(2b)$ and $\chi(u^2) = \chi(13a)$. Applying Proposition 5, we obtain:

$$\mu_{13}(u, \chi_3, *) = \tfrac{1}{26}(48\gamma_1) \geq 0; \quad \mu_1(u, \chi_3, *) = \tfrac{1}{26}(-4\gamma_1) \geq 0;$$
$$\mu_0(u, \chi_8, *) = \tfrac{1}{26}(24\gamma_2 + 28) \geq 0; \quad \mu_{13}(u, \chi_8, *) = \tfrac{1}{26}(-24\gamma_2 + 128) \geq 0.$$

It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{13a})$.

- $\chi(u^{13}) = \chi(2b)$ and $\chi(u^2) = \chi(13a)$. Applying Proposition 5, we obtain:

$$\mu_{13}(u, \chi_3, *) = \tfrac{1}{26}(48\gamma_1 + 48) \geq 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{26}(-48\gamma_1 + 56) \geq 0;$$
$$\mu_0(u, \chi_8, *) = \tfrac{1}{26}(24\gamma_2 + 76) \geq 0; \quad \mu_{13}(u, \chi_8, *) = \tfrac{1}{26}(-24\gamma_2 + 80) \geq 0;$$
$$\mu_1(u, \chi_{10}, *) = \tfrac{1}{26}(\gamma_3 + 303) \geq 0.$$

It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{13a})$.

- $\chi(u^{13}) = 3\chi(2a) - 2\chi(2b)$ and $\chi(u^2) = \chi(13a)$. Applying Proposition 5, we obtain:

$$\mu_2(u, \chi_3, *) = \tfrac{1}{26}(4\gamma_1 + 8) \geq 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{26}(-48\gamma_1 + 8) \geq 0;$$
$$\mu_0(u, \chi_8, *) = \tfrac{1}{26}(24\gamma_2 + 124) \geq 0; \quad \mu_{13}(u, \chi_8, *) = \tfrac{1}{26}(-24\gamma_2 + 32) \geq 0.$$

It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{13a})$.

Case $(viii)$. Let $u \in V(\mathbb{Z}G)$ where $|u| = 39$. Using Propositions 2 and 3,

$$\nu_{3a} + \nu_{13a} = 1.$$

Applying Proposition 5, we obtain the following system of inequalities:

$$\mu_{13}(u, \chi_3, *) = \tfrac{1}{39}(24\nu_{3a} + 54) \geq 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{39}(-48\nu_{3a} + 48) \geq 0;$$
$$\mu_1(u, \chi_3, *) = \tfrac{1}{39}(-2\nu_{3a} + 54) \geq 0.$$

Clearly $\nu_{3a} = 1$ and there are no possible integer solutions for (ν_{3a}, ν_{5a}).

Case (ix). Let $u \in V(\mathbb{Z}G)$ where $|u| = 65$. Using Propositions 2 and 3,

$$\nu_{5a} + \nu_{13a} = 1.$$

Applying Proposition 5, we obtain the following system of inequalities:

$$\mu_0(u, \chi_3, *) = \tfrac{1}{65}(96\nu_{5a} + 60) \geq 0; \quad \mu_{13}(u, \chi_3, *) = \tfrac{1}{65}(-24\nu_{5a} + 50) \geq 0;$$

Clearly there are no possible integer solutions for (ν_{5a}, ν_{13a}).

We shall now consider the prime graph of $G = {}^2F_4(2)$. Again, G contains elements of order 6 and 10. Therefore [2, 3] and [2, 5] are adjacent in $\pi(G)$ and consequently adjacent in $\pi(V(\mathbb{Z}G))$. Clearly $\pi(G) = \pi(V(\mathbb{Z}G))$, since there are no torsion units of order 15, 26, 39 and 65 in $V(\mathbb{Z}G)$. This completes the proof.

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